Examples of Matrix Multiplication

$$\begin{pmatrix} 2 & 5 & 1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

2×3 Can multiply 3×3

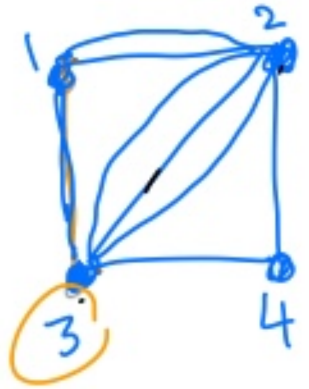
rows, # columns

$$= \begin{pmatrix} \underline{1^{st}\text{ row} \cdot 1^{st}\text{ column}} & \underline{1^{st}\text{ row} \cdot 2^{nd}\text{ col}} & \underline{1^{st}\text{ row} \cdot 3^{rd}\text{ col}} \\ \underline{2^{nd}\text{ row} \cdot 1^{st}\text{ col}} & \underline{2^{nd}\text{ row} \cdot 2^{nd}\text{ col}} & \underline{2^{nd}\text{ row} \cdot 3^{rd}\text{ col}} \end{pmatrix}$$

$$= \begin{pmatrix} \underline{2\cdot 1 + 5\cdot 0 + 1\cdot 0} & \underline{2\cdot 0 + 5\cdot 1 + 1\cdot 0} & \underline{2\cdot 0 + 5\cdot 1 + 1\cdot 1} \\ \underline{0\cdot 1 + (-1)(0) + 1(0)} & \underline{0\cdot 0 + (-1)\cdot 1 + (1)0} & \underline{0\cdot 0 + (-1)\cdot 1 + (1)\cdot 1} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 5 & 6 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$



$$A = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 3 & 1 \\ 1 & 3 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$1+1+0+1=3$
$=deg(3)$

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$$

If $1 \le i \le 4,\ 1 \le j \le 4$

A_{ij} = # of edges from i to j.

$A_{32} = 1$

one edge from vertex 3 to 2

$A^2 = (i\text{th row}) \cdot (j\text{th column})$

$$= \underbrace{\underbrace{A_{i1}}_{\#\text{ of edges from } i \text{ to } 1}\ \underbrace{A_{1j}}_{\#\text{ of edges from } 1 \text{ to } j}}_{\#\text{ of paths of length 2 from } i \text{ to } j \text{ when going through } \#1.} + A_{i2}A_{2j} + A_{i3}A_{3j} + A_{i4}A_{4j}$$

# of paths from i to j of length 2.

2 edges

$$\Rightarrow A^2 = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 3 & 1 \\ 1 & 3 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 3 & 1 \\ 1 & 3 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 3 & 1 \\ 1 & 3 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 3 & 6 & 3 \\ 3 & 14 & 3 & 3 \\ 6 & 3 & 11 & 3 \\ 3 & 3 & 3 & 2 \end{pmatrix}$$

$\Rightarrow$ There are 14 paths of length 2 from vertex 2 to itself.

There are 6 paths of length 2 from vertex 1 to 3.

$\Rightarrow$ The adjacency matrix A tells us

$$\underbrace{(A^n)_{ij}}_{\text{entry}} = \#\text{ of paths of length } n \text{ from } i \text{ to } j.$$

Note: Also a correct formula for directed graphs (paths must follow the directions.)

$\longrightarrow$ Question: How would I calculate the number of $\underbrace{\text{circuits}}_{\text{start \& end at same vertex}}$ of length n?

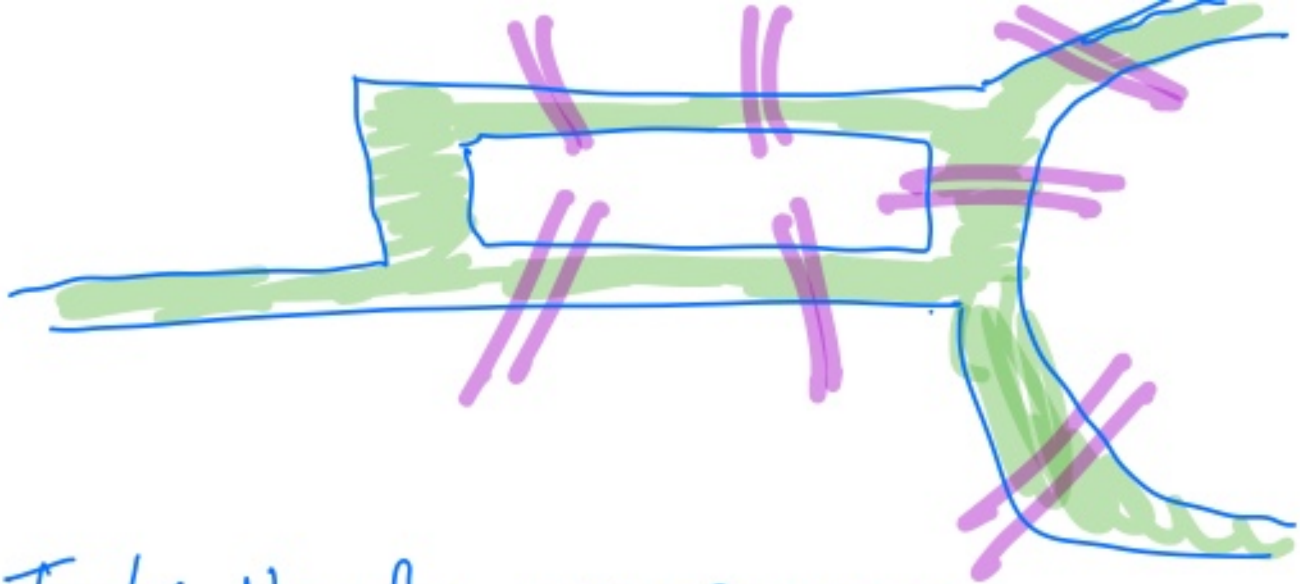
Answer: Look at $(A^n)_{ii}$ $\Leftarrow$ add over all i.

same vertex

i.e. add the diagonal entries of A^n.

Note: If we only care about the actual circuits and not the starting points, we are overcounting by a factor of n.

Important: Knowing # of paths of length n between vertices gives invariants of the graph — helps us tell when graphs are not isomorphic.



Task: Have fun run - go across each bridge once. How do we do it?.